Shear correction factors for beams, plates and shells

by

Fredy Andrés Mercado Navarro

NOTES

30 de julio de 2016

Structural elements are called *degenerate isoparametric elements* because in their formulation, the displacements u, v, w are interpolated in terms of midsurface displacements and rotations and because there is the major assumption that the stress normal to the midsurface is zero. Continuum elements, on the other hand, have their displacements interpolated in terms of nodal point displacements of the same kind.

Beam, plate and shell elements can be formulated using Bernoulli beam and Kirchhoff plate theory, in which shear deformations are neglected (taken from Finite Element Procedures, Bathe, p. 397).

Problem	**Displacem. component**	**Stress vector** τ^T	**Material matrix C**	**Strain vector** ϵ^T
Beam	w	$[M_{xx}]$	EI	$[\kappa_{xx}]$
Plate bending	w	$[M_{xx}\ M_{yy}\ M_{xy}]$	$\frac{Eh^3}{12(1-\nu^2)}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}$	$[\kappa_{xx}\ \kappa_{yy}\ \kappa_{xy}]$

Tabla 1: Kinematic and static variables in beam and plate bending problems.

With h being the thickness of the plate, I the moment of inertia and,

$$\kappa_{xx} = \frac{\partial^2 w}{\partial x^2} \qquad \kappa_{yy} = \frac{\partial^2 w}{\partial y^2} \qquad \kappa_{xy} = 2\frac{\partial^2 w}{\partial x \partial y} \tag{1}$$

1. Bernoulli beam theory (excludes shear deformations)

Also called Clasical beam theory. The basic assumption in beam bending analysis exluding shear deformations is that a normal to the midsurface (neutral axis) of the beam remains straight during deformation and that its angular rotation is equal to the slope of the beam midsurface (see Figure 1).

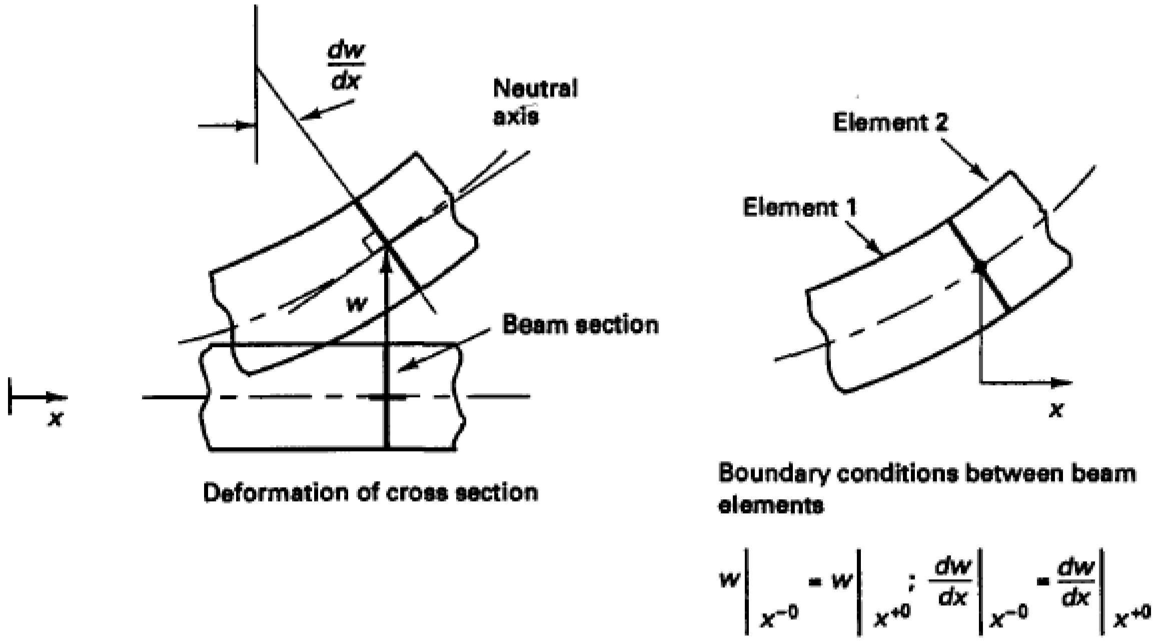


Figura 1: Beam deformations excluding shear effect.

This kinematic assumption leads to the well-known beam-bending governing differential equation in which the transverse displacement w is the only variable. Therefore, using beam elements formulated with this theory, displacement continuity between elements requires that w and dw/dx be continuous.

$$\beta_{\text{Bernoulli}} = \beta(x) = \frac{dw}{dx} \tag{2}$$

NOTE: don't forget that w is the displacement in z direction.

Following the assumptions, the displacement field is written as,

$$u = -z\beta(x) = -z\frac{dw}{dx} \qquad v = 0 \qquad w = w(x) \tag{3}$$

The procedure to derive u is (see Fig. 2 and Oñate, vol. 2),

$$\sin\beta = \frac{u}{z} \qquad \beta \to 0 \qquad \beta = \frac{u}{z} \qquad u = -z\beta \tag{4}$$

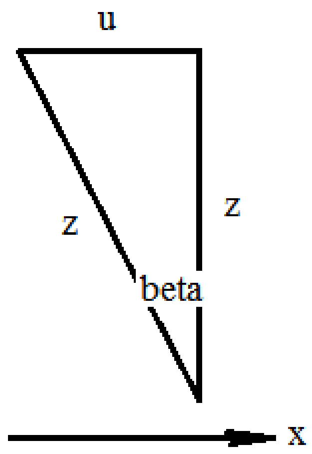


Figura 2: Derivation of u.

Rotation is equal to the slope of the beam axis. Strain and stress fields are,

$$\varepsilon_x = \frac{du}{dx} = -z\frac{d^2w}{dx^2} = -z\kappa \qquad \varepsilon_y = \varepsilon_z = \gamma_{xy} = \gamma_{xz} = \gamma_{yz} = 0 \tag{5}$$

i.e. the strain is under a pure axial strain (ε_x) state. The axial stress σ_x is related to ε_x by Hook law as,

$$\sigma_x = E\varepsilon_x = -zE\frac{d^2w}{dx^2} \tag{6}$$

E is the Young modulus. The bending moment for a cross section is defined as (Oñate Vol. 2, pág. 3),

$$M = -\int\int_A z\sigma_x dA = \left(\int\int_A z^2 dA\right) E\frac{d^2w}{dx^2} = EI_y\frac{d^2w}{dx^2} = EI_y\kappa \tag{7}$$

where,

$$I_y = \left(\int\int_A z^2 dA\right) \tag{8}$$

is the moment of inertia of the section with respect to the y axis and,

$$\kappa = \frac{d^2w}{dx^2} \tag{9}$$

is the curvature of the beam axis.

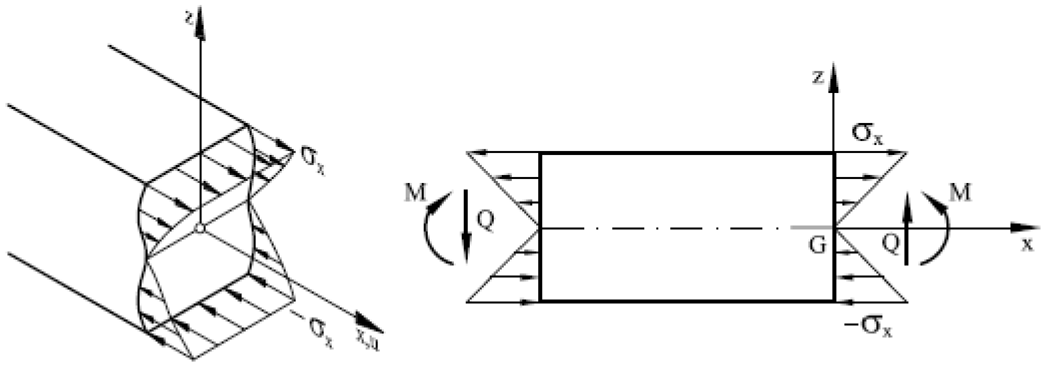


Figura 3: Sign criteria for axial stress σ_x, bending moment M and shear force Q.

2. Timoshenko beam theory (includes shear deformations)

This theory is also called Thick beam theory. Here we retain the assumption that a plane section originally normal to the neutral axis remains plane, but because of shear deformations this section does not remain normal to the neutral axis (see Figure 4). Timoshenko hypothesis is equivalent to assuming an average rotation for the deformed cross section which is kept plane (Oñate, Vol. 2, p. 38).

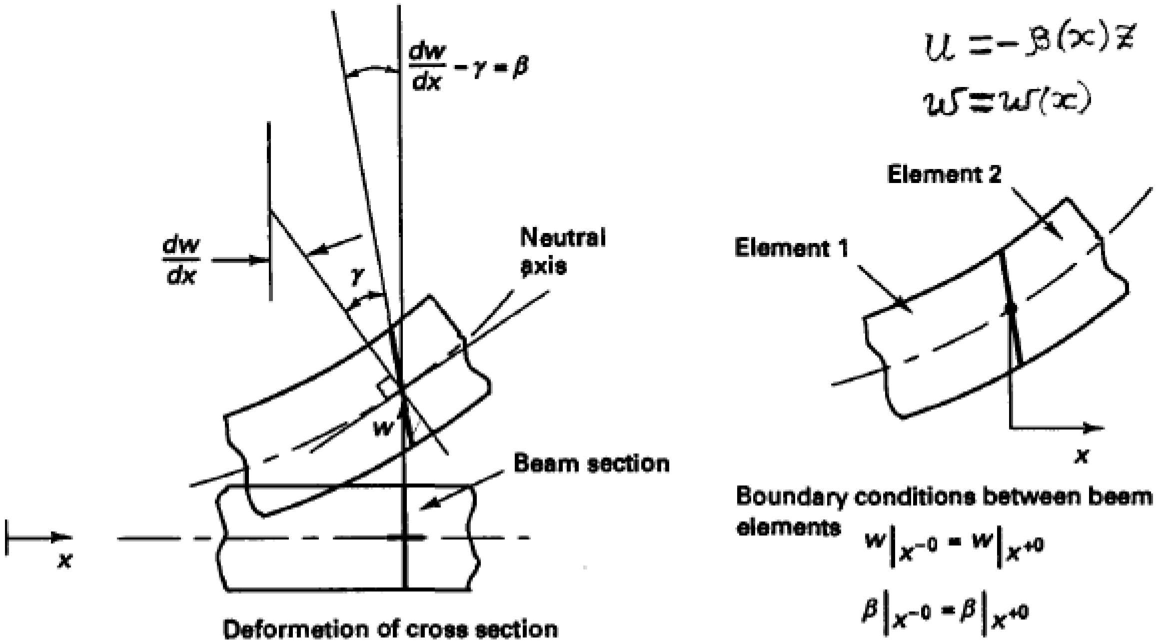


Figura 4: Beam deformations including shear effect.

The total rotation of the plane originally normal to the neutral axis of the beam is given by the rotation of the tangent to the neutral axis and the shear deformation,

$$\beta_{\text{Timoshenko}} = \frac{dw}{dx} - \gamma \tag{10}$$

So, $\beta(x)$ is the angle with respect to the z axis that the shear strains produce in the material, dw/dx is the slope of the beam axis and γ is a **constant shearing strain** across the section (or an additional rotation due to the distortion of the cross-section). The displacement field is written as,

$$u = -z\beta(x) = -z\left(\frac{dw}{dx} - \gamma\right) \qquad v = 0 \qquad w = w(x) \tag{11}$$

The strain and stress fields are (use Eq. 11 to derive the expressions),

$$\varepsilon_x = \frac{du}{dx} = -z\frac{d\beta}{dx} \qquad \gamma_{xz}(x) = \frac{dw}{dx} + \frac{du}{dz} = \frac{dw(x)}{dx} - \beta(x) = \gamma \tag{12}$$

The other strain components are zero. Hence, Timoshenko theory introduces a transverse shear deformation γ_{xz}, which value coincides with the rotation γ (see Fig. 4). The axial and shear stresses σ_x and τ_{xz} at a point of the beam cross section are related to the corresponding strains by,

$$\sigma_x(x,z) = E\varepsilon_x = -zE\frac{d\beta(x)}{dx} \qquad \tau_{xz}(x) = G\gamma_{xz}(x) = G\left(\frac{dw(x)}{dx} - \beta(x)\right) \tag{13}$$

where G is the shear modulus,

$$G = \frac{E}{2(1+\nu)} \tag{14}$$

and ν is the Poisson ratio. Timoshenko beam theory accounts for the effect of transverse shear deformation. Timoshenko beam elements are therefore applicable for “thick” beams ($L/h < 10$) where transverse shear deformation has an influence in the solution, as well as for slender (esbeltas) beams ($L/h > 100$) where the influence is irrelevant (Oñate Vol. 2, pág. 37).

2.1. Timoshenko: resultant stresses and generalized strains

The bending moment M and the shear force Q are defined with the sign criterion of Figure 5 as,

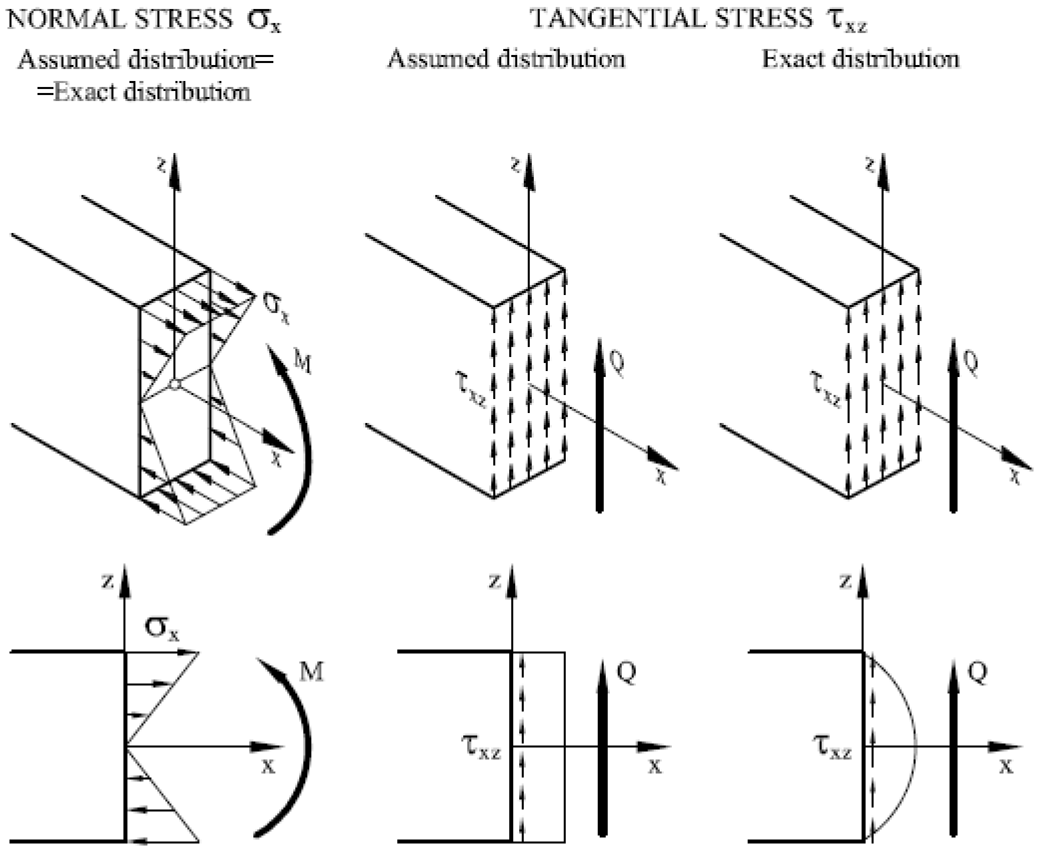


Figura 5: Timoshenko beam. Normal and tangential stresses.

$$M = -\int\int_A z\sigma_x dA \qquad Q = \int\int_A \tau_{xz} dA \tag{15}$$

Remember that $\sigma_x = \sigma_x(x,z)$ and $\tau_{xz} = \tau_{xz}(x)$. This means that if we fix the x coordinate the stress σ_x will vary linearly with z. On the other hand τ_{xz} is just a function of the x coordinate, so it is constant through the thickness of the beam. Substituting Eq. 13 in Eq. 15 we get,

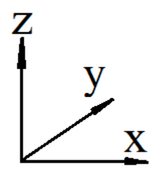
$$M = -\int\int_A z\left(-zE\frac{d\beta(x)}{dx}\right) dA \qquad Q = \int\int_A \left(G\gamma_{xz}(x)\right) dA \tag{16}$$

$$M = \left[\int\int_A Ez^2 dA\right] \cdot \frac{d\beta(x)}{dx} \qquad Q = \left[\int\int_A G dA\right] \cdot \gamma_{xz}(x) \tag{17}$$

$$M = \hat{D}_{\text{bending}} \cdot \frac{d\beta(x)}{dx} \qquad Q = \hat{G} \cdot \gamma_{xz}(x) \tag{18}$$

$\kappa = d\beta(x)/dx$ is the bending strain (not curvature). $\hat{D}_{\text{bending}}$ and $\hat{G}$ are resultant values over the section. For homogeneous materials,

$$\hat{D}_{\text{bending}} = \int\int_A Ez^2 dA = E\left[\int\int_A z^2 dA\right] = EI_y \qquad \hat{G} = GA \qquad (19)$$



σ_x varies linearly through the thickness, which is considered "exact" according to **classical beam theory**, but τ_{xz} is constant across the thickness, which is in contradiction with the exact quadratic distribution for a rectangular beam.

2.2. Shear stress distribution in rectangular beam cross section

The shear stress analysis presented here was developed by D. J. Jourawski (russian engineer, 1821-1891. See Gere). The Eq. for the shear stress over the cross section that comes from force equilibrium equations is (see Strength of materials by Timoshenko, Vol. 1, page 114),

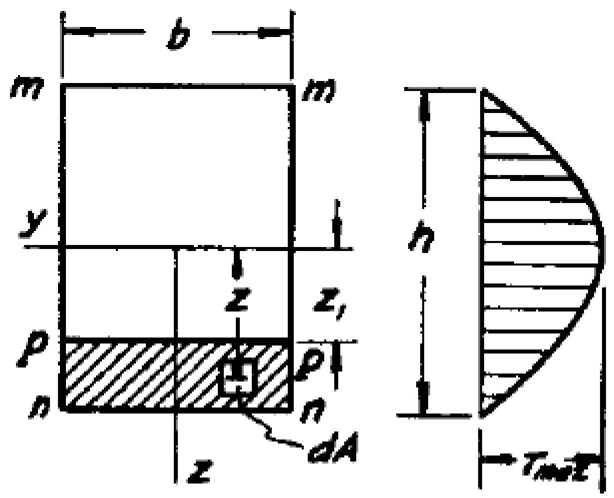


Figura 6: Figure taken from book of Timoshenko, Vol. 1.

$$\tau_{xz} = \frac{V}{I_y b}\int_{z_1}^{h/2} z dA \qquad (20)$$

V is the shear force, I_y is the moment of inertia around y axis and b is the width of the cross section. The following integral is the moment of the shaded portion of the cross section with respect to the neutral axis y (with $dA = bdz$ for rectangular cross section).

$$\int_{z_1}^{h/2} z dA = \frac{b}{2}\left[\frac{h^2}{4} - z_1^2\right] \tag{21}$$

the same result can be obtained by multiplying the area of the shaded portion

$$b\left[\frac{h}{2} - y_1\right] \quad \text{by the distance} \quad \frac{1}{2}\left[\frac{h}{2} + z_1\right] \tag{22}$$

Which is the distance of its centroid to the neutral axis. Substituting Eq. 21 into Eq. 20 we get,

$$\tau_{xz} = \frac{V}{2I_y}\left[\frac{h^2}{4} - z_1^2\right] \tag{23}$$

Shearing stresses are not uniformly distributed from top to bottom of the beam. The maximum value of τ_{xz} occurs for $z_1 = 0$ (on the neutral axis). If the moment of inertia around y is,

$$I_y = \frac{bh^3}{12} \tag{24}$$

Then Eq. 23 becomes,

$$\tau_{xz}(z_1) = \frac{3V}{2A}\left[\frac{\left(\frac{h}{2}\right)^2 - z_1^2}{\left(\frac{h}{2}\right)^2}\right] = \frac{3V}{2A}\left[1 - 4\frac{z_1^2}{h^2}\right] \quad \text{for} \quad -\frac{h}{2} \leq z_1 \leq \frac{h}{2} \tag{25}$$

Maximum shear stress occurs for $z_1 = 0$,

$$(\tau_{xz})_{\text{máx}} = \frac{3V}{2A} \tag{26}$$

For the bottom and for the top of the cross section $z_1 = \pm h/2$ and $\tau_{xz} = 0$.

2.3. Shear strain energy

The strain energy in a general state of stress and strain is,

$$dU = \frac{1}{2}\left(\sigma_{xx}\varepsilon_{xx} + ... + \tau_{xy}\gamma_{xy} + ... + \tau_{xz}\gamma_{xz}\right) dV \tag{27}$$

Integrating over the entire volume,

$$U = \frac{1}{2}\int_V \left(\sigma_{xx}\varepsilon_{xx} + ... + \tau_{xy}\gamma_{xy} + ... + \tau_{xz}\gamma_{xz}\right) dV \tag{28}$$

$$U = \frac{1}{2}\int_V \underline{\sigma}^T \underline{\varepsilon} dV \tag{29}$$

The shear strain energy for the beam is,

$$U_s = \frac{1}{2}\int_V \tau_{xz}\gamma_{xz} dV \quad \text{with} \quad \gamma_{xz} = \frac{\tau_{xz}}{G_{xz}} \tag{30}$$

then,

$$U_s = \frac{1}{2G_{xz}} \int_V \tau_{xz}^2 dV \tag{31}$$

Using τ_{xz} found before for a rectangular cross section where shear stresses have a parabolic distribution we have,

$$U_s = \frac{1}{2G_{xz}} \int_{-h/2}^{h/2} \left[\frac{3V}{2bh}\left(1 - 4\frac{z^2}{h^2}\right)\right]^2 \int_0^b \int_0^L dxdydz \tag{32}$$

$$U_s = \frac{bL}{2G_{xz}} \int_{-h/2}^{h/2} \left[\frac{3V}{2bh}\left(1 - 4\frac{z^2}{h^2}\right)\right]^2 dz \tag{33}$$

$$U_s = \frac{1}{2} \cdot \frac{9}{4} \frac{bLV^2}{h^2b^2G_{xz}} \int_{-h/2}^{h/2} \left(1 - \frac{8}{h^2}z^2 + \frac{16}{h^4}z^4\right) dz \tag{34}$$

$$U_s = \frac{1}{2} \cdot \frac{9}{4} \frac{bLV^2}{h^2b^2G_{xz}} \left[\int_{-h/2}^{h/2} dz - \frac{8}{h^2}\int_{-h/2}^{h/2} z^2dz + \frac{16}{h^4}\int_{-h/2}^{h/2} z^4dz\right] \tag{35}$$

$$U_s = \frac{1}{2} \cdot \frac{9}{4} \frac{bLV^2}{h^2b^2G_{xz}} \left[h - \frac{2}{3}h + \frac{1}{5}h\right] \tag{36}$$

$$U_s = \frac{1}{2} \cdot \frac{9}{4} \frac{bLV^2}{h^2b^2G_{xz}} \left[\frac{8}{15}h\right] = \frac{1}{2} \cdot \frac{3}{1} \frac{LV^2}{hbG_{xz}} \left[\frac{2}{5}\right] \tag{37}$$

$$U_{s,\text{parabolic}} = \frac{1}{2} \cdot \frac{LV^2}{hbG_{xz}} \cdot \left[\frac{6}{5}\right] \tag{38}$$

Now, assuming a constant shear distribution we have $\tau_{xz} = V/A$, then based in Eq. 31,

$$U_{s,\text{constant}} = \frac{1}{2G_{xz}} \int_V \tau_{xz}^2 dV = \frac{1}{2G_{xz}} \int_V \left(\frac{V}{A}\right)^2 dV \tag{39}$$

$$U_{s,\text{constant}} = \frac{V^2}{2h^2b^2G_{xz}} \int_{-h/2}^{h/2} dz \int_0^b dy \int_0^L dx = \frac{hbLV^2}{2h^2b^2G_{xz}} \tag{40}$$

$$U_{s,\text{constant}} = \frac{LV^2}{2hbG_{xz}} \tag{41}$$

Strain energies are made equal and a shear correction factor κ_{xz} is introduced in the expression for the strain energy given by the constant shear stress distribution,

$$U_{s,\text{constant}} = U_{s,\text{parabolic}} \tag{42}$$

$$\frac{LV^2}{2hb\kappa_{xz}G_{xz}} = \frac{1}{2} \cdot \frac{LV^2}{hbG_{xz}} \cdot \left[\frac{6}{5}\right] \tag{43}$$

$$\frac{LV^2}{2hb\left(\kappa_{xz}G_{xz}\right)} = \frac{1}{2} \cdot \frac{LV^2}{hb\left(\frac{5}{6}G_{xz}\right)} \tag{44}$$

Finally,

$$\kappa_{xz} = \frac{5}{6} \tag{45}$$

See http://people.duke.edu/ hpgavin/cee201/strain-energy.pdf for more interesting information.

Since the actual shearing stress and strain vary over the section, the shearing strain γ in Eq. 10 is an **equivalent constant strain** on a corresponding shear area A_s,

$$\tau = \frac{V}{A_s} \qquad \gamma = \frac{\tau}{G} \qquad \kappa = \frac{A_s}{A} \tag{46}$$

where V is the shear force at the section being considered. **Different assumptions** may be used to evaluate a reasonable factor κ. **One simple procedure** is to evaluate the SCF using the condition that when action on A_s, the constant shear stress in Eq. 46 must yield the same shear strain energy as the actual shearing stress (evaluated from beam theory) acting on the actual cross-sectional area A of the beam.

3. Plate bending

To calculate the generalized stress-strain matrix for plate bending analysis the stress-strain matrix corresponding to plane stress conditions is used. The strains at a distance z measured upward from the midsurface of the plate are,

$$\begin{bmatrix} -z\frac{\partial^2 w}{\partial x^2} & -z\frac{\partial^2 w}{\partial y^2} & -2z\frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \tag{47}$$

In plate bending analysis it is assumed that each layer of the plate acts in plane stress condition and positive curvatures correspond to positive moments. Hence, integrating the normal stresses in the plate to obtain moments per unit length, the generalized stress-strain matrix is,

$$C_b = \int_{-h/2}^{+h/2} z^2 \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} dz \tag{48}$$

We have that,

$$\int_{-h/2}^{+h/2} z^2 dz = \frac{h^3}{12} \tag{49}$$

Then,

$$C_b = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{50}$$

The basic proposition in plate bending and shell analyses is that the structure is thin in one dimension, and therefore the following **assumptions** can be made:

1. The stress through the thickness (i.e., perpendicular to the midsurface) of the plate/shell is zero.

2. Material particles that are originally on a straight line perpendicullar to the midsurface of the plate/shell remain on a straight line during deformations.

KIRCHHOFF THEORY: shear deformations are neglected (the straight line remains perpendicular to the midsurface during deformations).

REISSNER/MINDLIN THEORY: shear deformations are included (the straight line in general does not remain perpendicular to the midsurface during deformations).

4. Kirchhoff plate theory (excludes shear deformations)

For the small displacement bending theory. Rotations in the x,z and y,z planes:

$$\beta_x = \frac{\partial w}{\partial x} \qquad \beta_y = \frac{\partial w}{\partial y} \tag{51}$$

5. Reissner-Mindlin plate theory (includes shear deformations)

For the small displacement bending theory. Displacements are:

$$u = -z\beta_x(x,y) \qquad v = -z\beta_y(x,y) \qquad w = w(x,y) \tag{52}$$

with rotations as,

$$\beta_x = \frac{\partial w}{\partial x} \qquad \beta_y = \frac{\partial w}{\partial y} \tag{53}$$

See Figure 7. For this case bending strains ε_{xx}, ε_{yy} and ε_{xy} vary linearly through the plate thickness and are given by the curvatures of the plate using Ec. 52,

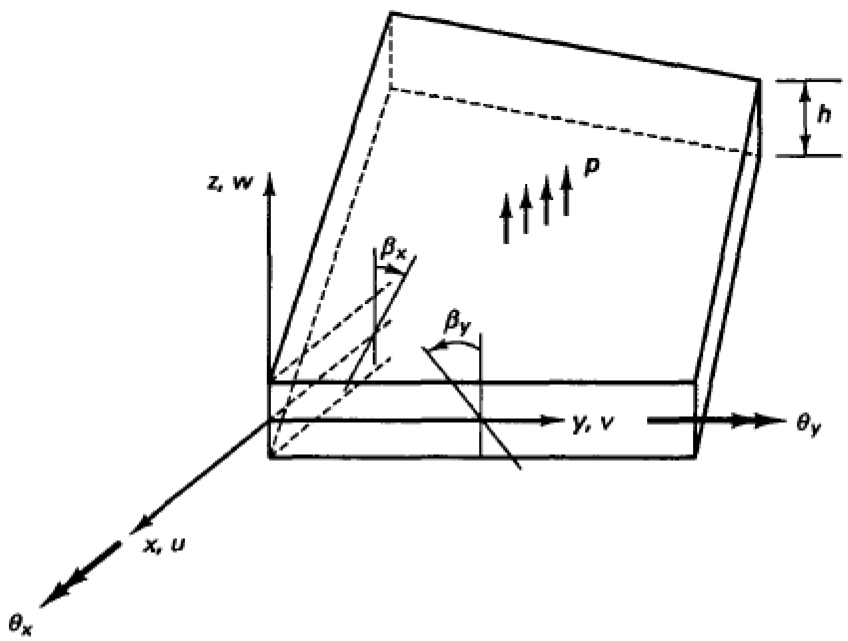


Figura 7: Deformation assumptions in analysis of plates including shear deformations.

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[-z\beta_x(x,y)\right] = -z\frac{\partial \beta_x(x,y)}{\partial x} \tag{54}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left[-z\beta_y(x,y)\right] = -z\frac{\partial \beta_y(x,y)}{\partial y} \tag{55}$$

$$\gamma_{xy} = 2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial}{\partial y}\left(-z\beta_x\right) + \frac{\partial}{\partial x}\left(-z\beta_y\right) = -z\left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right) \tag{56}$$

then, in matrix form,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = -z \begin{bmatrix} \dfrac{\partial \beta_x}{\partial x} \\ \dfrac{\partial \beta_y}{\partial y} \\ \dfrac{\partial \beta_x}{\partial y} + \dfrac{\partial \beta_y}{\partial x} \end{bmatrix} \tag{57}$$

whereas the transverse shear strains are assumed to be constant through the thickness of the plate,

$$\begin{bmatrix} \varepsilon_{xz} \\ \varepsilon_{yz} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial w}{\partial x} - \beta_x \\ \dfrac{\partial w}{\partial y} - \beta_y \end{bmatrix} \tag{58}$$

Each transverse shear strain component is of the form used in the description of the beam deformations.