Constitutive relation for plane stress with transverse shear behavior

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17 de julio de 2016

The fourth-rank tensor of interest is the elastic stiffness matrix, which appears in the generalized Hooke's law as follows:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{1}$$

Stress and strain tensors have 9 components in 3-space and the constitutive tensor 81.

$$\sigma_{11} = C_{1111}\varepsilon_{11} + C_{1112}\varepsilon_{12} + C_{1113}\varepsilon_{13} + C_{1121}\varepsilon_{21} + C_{1122}\varepsilon_{22} + C_{1123}\varepsilon_{23} + C_{1131}\varepsilon_{31} + C_{1132}\varepsilon_{32} + C_{1133}\varepsilon_{33} \tag{2}$$

$$\sigma_{12} = C_{1211}\varepsilon_{11} + C_{1212}\varepsilon_{12} + C_{1213}\varepsilon_{13} + C_{1221}\varepsilon_{21} + C_{1222}\varepsilon_{22} + C_{1223}\varepsilon_{23} + C_{1231}\varepsilon_{31} + C_{1232}\varepsilon_{32} + C_{1233}\varepsilon_{33} \tag{3}$$

$$\sigma_{13} = C_{1311}\varepsilon_{11} + C_{1312}\varepsilon_{12} + C_{1313}\varepsilon_{13} + C_{1321}\varepsilon_{21} + C_{1322}\varepsilon_{22} + C_{1323}\varepsilon_{23} + C_{1331}\varepsilon_{31} + C_{1332}\varepsilon_{32} + C_{1333}\varepsilon_{33} \tag{4}$$

$$\sigma_{21} = C_{2111}\varepsilon_{11} + C_{2112}\varepsilon_{12} + C_{2113}\varepsilon_{13} + C_{2121}\varepsilon_{21} + C_{2122}\varepsilon_{22} + C_{2123}\varepsilon_{23} + C_{2131}\varepsilon_{31} + C_{2132}\varepsilon_{32} + C_{2133}\varepsilon_{33} \tag{5}$$

$$\sigma_{22} = C_{2211}\varepsilon_{11} + C_{2212}\varepsilon_{12} + C_{2213}\varepsilon_{13} + C_{2221}\varepsilon_{21} + C_{2222}\varepsilon_{22} + C_{2223}\varepsilon_{23} + C_{2231}\varepsilon_{31} + C_{2232}\varepsilon_{32} + C_{2233}\varepsilon_{33} \tag{6}$$

$$\sigma_{23} = C_{2311}\varepsilon_{11} + C_{2312}\varepsilon_{12} + C_{2313}\varepsilon_{13} + C_{2321}\varepsilon_{21} + C_{2322}\varepsilon_{22} + C_{2323}\varepsilon_{23} + C_{2331}\varepsilon_{31} + C_{2332}\varepsilon_{32} + C_{2333}\varepsilon_{33} \tag{7}$$

$$\sigma_{31} = C_{3111}\varepsilon_{11} + C_{3112}\varepsilon_{12} + C_{3113}\varepsilon_{13} + C_{3121}\varepsilon_{21} + C_{3122}\varepsilon_{22} + C_{3123}\varepsilon_{23} + C_{3131}\varepsilon_{31} + C_{3132}\varepsilon_{32} + C_{3133}\varepsilon_{33} \tag{8}$$

$$\sigma_{32} = C_{3211}\varepsilon_{11} + C_{3212}\varepsilon_{12} + C_{3213}\varepsilon_{13} + C_{3221}\varepsilon_{21} + C_{3222}\varepsilon_{22} + C_{3223}\varepsilon_{23} + C_{3231}\varepsilon_{31} + C_{3232}\varepsilon_{32} + C_{3233}\varepsilon_{33} \tag{9}$$

$$\sigma_{33} = C_{3311}\varepsilon_{11} + C_{3312}\varepsilon_{12} + C_{3313}\varepsilon_{13} + C_{3321}\varepsilon_{21} + C_{3322}\varepsilon_{22} + C_{3323}\varepsilon_{23} + C_{3331}\varepsilon_{31} + C_{3332}\varepsilon_{32} + C_{3333}\varepsilon_{33} \tag{10}$$

If stress and strain tensors are symmetric i.e.,

$$\sigma_{ij} = \sigma_{ji} \quad \text{and} \quad \varepsilon_{ij} = \varepsilon_{ji} \tag{11}$$

We can consider first symmetry of strain to get a reduction of three constants for $C_{ijkl} = C_{ijlk}$ for each index i and j which gives us a reduction of $9 \times 3 = 27$, leaving 54 independent constants. Next we have symmetry of stress which gives $C_{jikl} = C_{ijkl}$. In this case we have again 27 potential constants that are reduced, since there are three duplicate ij constants for nine kl constants. However, there is an overlap with C_{3131}, C_{3232}, C_{2121}, C_{3113}, C_{3223}, C_{2112}, C_{1331}, C_{2332}, C_{1221}, leaving us to reduce the number of independent constants by 18. This leaves us with 36 independent constants.

We have major symmetry due to the existence of a strain energy function. The following expression is the strain energy in a linear elastic material (for small deformations, Cauchy stress and infinitesimal strains),

$$W = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \tag{12}$$

Cauchy stress was defined using Hooke's law, so we can write the strain energy as presented in the second part of Eq. 12. If we differentiate the strain energy function twice with respect to the strain we obtain the Hooke's law constants:

$$\frac{\partial W}{\partial \varepsilon_{ij}} = C_{ijkl} \varepsilon_{kl} = \sigma_{ij} \tag{13}$$

Differentiating once more we get,

$$\frac{\partial^2 W}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} = C_{ijkl} \tag{14}$$

Similarly,

$$\frac{\partial^2 W}{\partial \varepsilon_{kl} \partial \varepsilon_{ij}} = C_{klij} \tag{15}$$

but the order of differentiation of W is immaterial, so,

$$C_{ijkl} = C_{klij} \tag{16}$$

This allows us to reduce the number of independent constants by 15 to 21. Thus, for any linear elastic material we need to measure at most 21 constants. This last portion of the text was taken from http://www.umich.edu/ bme332/ch6consteqelasticity/bme332consteqelasticity.htm (course BME 332: Introduction to Biosolid Mechanics). I also took information from Mechanics of composite materials, second Ed. (Jones, 1999).

0.1. Defining constitutive relation for full anisotropic elasticity

A linear elastic material model is valid for small elastic strains (normally less than 5 %. See Abaqus 6.14 Analysis user's guide on internet) and can have properties that depend on temperature or other field variables. For the general case of anisotropic elasticity the stress-strain relationship can be expressed as a 6x6 matrix. Anisotropic materials have no symmetry planes for the elastic properties. The constitutive relation for linear elastic anisotropic materials is,

$$C = \begin{bmatrix} C^{1111} & C^{1122} & C^{1133} & C^{1112} & C^{1123} & C^{1113} \\ C^{2211} & C^{2222} & C^{2233} & C^{2212} & C^{2223} & C^{2213} \\ C^{3311} & C^{3322} & C^{3333} & C^{3312} & C^{3323} & C^{3313} \\ C^{1211} & C^{1222} & C^{1233} & C^{1212} & C^{1223} & C^{1213} \\ C^{2311} & C^{2322} & C^{2333} & C^{2312} & C^{2323} & C^{2313} \\ C^{1311} & C^{1322} & C^{1333} & C^{1312} & C^{1323} & C^{1313} \end{bmatrix} \tag{17}$$

If its symmetry is taken into consideration we have only 21 independent elastic stiffness parameters,

$$C = \begin{bmatrix} C^{1111} & C^{1122} & C^{1133} & C^{1112} & C^{1123} & C^{1113} \\ & C^{2222} & C^{2233} & C^{2212} & C^{2223} & C^{2213} \\ & & C^{3333} & C^{3312} & C^{3323} & C^{3313} \\ & & & C^{1212} & C^{1223} & C^{1213} \\ & sym & & & C^{2323} & C^{2313} \\ & & & & & C^{1313} \end{bmatrix} \quad (18)$$

For the linear elastic material model the Cauchy stresses are defined in terms of the constitutive model and the infinitesimal strains as,

$$\underline{\sigma} = \underline{\underline{C}} \cdot \underline{\varepsilon} \quad (19)$$

0.1.1. Stability

The restrictions imposed upon the elastic constants by stability requirements are too complex to express in terms of simple equations. However, the requirement that C is positive definite requires that all of the eigenvalues of the elasticity matrix be positive.

0.2. Linear elastic orthotropic material

Orthotropic materials have three orthogonal symmetry planes for the elastic properties passing through every point in the material. Strain expressions for the most general linear elastic orthotropic material are,

$$\varepsilon_1 = \frac{1}{E_1}\sigma_1 - \frac{\nu_{21}}{E_2}\sigma_2 - \frac{\nu_{31}}{E_3}\sigma_3 \quad (20)$$

$$\varepsilon_2 = \frac{1}{E_2}\sigma_2 - \frac{\nu_{12}}{E_1}\sigma_1 - \frac{\nu_{32}}{E_3}\sigma_3 \quad (21)$$

$$\varepsilon_3 = \frac{1}{E_3}\sigma_3 - \frac{\nu_{13}}{E_1}\sigma_1 - \frac{\nu_{23}}{E_2}\sigma_2 \quad (22)$$

$$\gamma_{12} = \frac{\tau_{12}}{G_{12}} \quad (23)$$

$$\gamma_{13} = \frac{\tau_{13}}{G_{13}} \quad (24)$$

$$\gamma_{23} = \frac{\tau_{23}}{G_{23}} \quad (25)$$

Now we can form the most general expression for the compliance matrix in 3D orthotropic elasticity:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{23}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{13}} \end{bmatrix} \cdot \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{23} \\ \tau_{13} \end{bmatrix} \tag{26}$$

0.2.1. Poisson's ratio

ν_{ij} means transverse strain in the j-direction when the material is stressed in the i-direction and,

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \tag{27}$$

0.2.2. Stability

The conditions of material stability or Drucker stability must be satisfied. These conditions require that the tensor C be positive definite, which leads to certain restrictions on the values of the elastic constants.

$$E_1,\ E_2,\ E_3,\ G_{12},\ G_{23},\ G_{13} > 0 \tag{28}$$

$$|\nu_{12}| < \left(\frac{E_1}{E_2}\right)^{1/2} \qquad |\nu_{13}| < \left(\frac{E_1}{E_3}\right)^{1/2} \qquad |\nu_{23}| < \left(\frac{E_2}{E_3}\right)^{1/2} \tag{29}$$

Replacing ν_{ij} from Eq. 27 into the expressions on Eq. 29 and operating we can get,

$$|\nu_{21}| < \left(\frac{E_2}{E_1}\right)^{1/2} \qquad |\nu_{31}| < \left(\frac{E_3}{E_1}\right)^{1/2} \qquad |\nu_{32}| < \left(\frac{E_3}{E_2}\right)^{1/2} \tag{30}$$

$$1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{32}\nu_{13} > 0 \tag{31}$$

When the left-hand side of the inequality approaches zero, the material exhibits incompressible behavior.

0.3. Plane stress with transverse shear material model

This material model is used for plates and shells. If we assume $\sigma_3 = 0$ in Eq. 26 we get,

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{23}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{13}} \end{bmatrix} \cdot \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ 0 \\ \tau_{12} \\ \tau_{23} \\ \tau_{13} \end{bmatrix} \tag{32}$$

This is how we get the compliance matrix for orthotropic elasticity in plane stress with transverse shear deformation,

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{G_{12}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} \end{bmatrix} \cdot \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \\ \tau_{23} \\ \tau_{13} \end{bmatrix} \tag{33}$$

We can't forget that strains can still be calculated as,

$$\varepsilon_3 = -\frac{\nu_{13}}{E_1}\sigma_1 - \frac{\nu_{23}}{E_2}\sigma_2 \tag{34}$$

Now we are going to find the constitutive relation. Taking into consideration Eq. 33 we can do,

$$\sigma_1 = E_1\varepsilon_1 + \frac{\nu_{21}E_1}{E_2}\sigma_2 \tag{35}$$

Substituting σ_1 in the expression for ε_2 from Eq. 33 we get,

$$\sigma_2 = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}\varepsilon_1 + \frac{E_2}{1-\nu_{12}\nu_{21}}\varepsilon_2 \tag{36}$$

Now we can substitute last σ_2 in Eq. 35 and we get,

$$\sigma_1 = \frac{E_1}{1-\nu_{12}\nu_{21}}\varepsilon_1 + \frac{\nu_{21}E_1}{1-\nu_{21}\nu_{12}}\varepsilon_2 \tag{37}$$

With $D = 1 - \nu_{12}\nu_{21}$ we have,

$$\sigma_1 = \frac{E_1}{D}\varepsilon_1 + \frac{\nu_{21}E_1}{D}\varepsilon_2 \tag{38}$$

$$\sigma_2 = \frac{\nu_{12}E_2}{D}\varepsilon_1 + \frac{E_2}{D}\varepsilon_2 \tag{39}$$

$$\tau_{12} = G_{12}\gamma_{12} \qquad \tau_{23} = G_{23}\gamma_{23} \qquad \tau_{31} = G_{31}\gamma_{31} \tag{40}$$

Finally, we get,

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \\ \tau_{23} \\ \tau_{13} \end{bmatrix} = \begin{bmatrix} \frac{E_1}{D} & \frac{\nu_{21} E_1}{D} & 0 & 0 & 0 \\ \frac{\nu_{12} E_2}{D} & \frac{E_2}{D} & 0 & 0 & 0 \\ 0 & 0 & G_{12} & 0 & 0 \\ 0 & 0 & 0 & G_{23} & 0 \\ 0 & 0 & 0 & 0 & G_{13} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{bmatrix} \tag{41}$$

Again with,

$$\varepsilon_3 = -\frac{\nu_{13}}{E_1}\sigma_1 - \frac{\nu_{23}}{E_2}\sigma_2 \tag{42}$$

Due to the symmetry we have $C_{12} = C_{21}$, then,

$$\nu_{21} E_1 = \nu_{12} E_2 \tag{43}$$

With this material model for orthotropic elasticity in plane stress (with transverse shear deformation included) we just need six values to define an orthotropic material: E_1, E_2, ν_{12}, G_{12}, $G_2 3$ and G_{13} because Poisson's ratio ν_{21} can be calculated from Eq. 43 as,

$$\nu_{21} = \frac{E_2}{E_1}\nu_{12} \tag{44}$$

0.3.1. Stability

Material stability for planes stress requires (see Abaqus 6.14 Analysis User's Guide),

$$E_1,\ E_2,\ G_{12},\ G_{23},\ G_{13} > 0 \tag{45}$$

$$|\nu_{12}| < \left(\frac{E_1}{E_2}\right)^{1/2} \tag{46}$$

0.4. Isotropic materials

Have an infinite number of symmetry planes passing through every point in the material.

0.4.1. Stability

The stability criterion requieres that $E > 0$, $G > 0$ and $-1 < \nu < 0{,}5$. Values of $\nu \to 0{,}5$ result in nearly incompressible behavior.